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Index theorem for topological excitations on $\mathrm{R}^3 \times \mathrm{S}^1$ and Chern-Simons theory

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Abstract: We derive an index theorem for the Dirac operator in the background of various topological excitations on an $R^3 \times S^1$ geometry. The index theorem provides more refined data than the APS index for an instanton on R^4 and reproduces it in decompactification limit. In the R^3 limit, it reduces to the Callias index theorem. The index is expressed in terms of topological charge and the η-invariant associated with the boundary Dirac operator. Neither topological charge nor η-invariant is typically an integer, however, the non-integer parts cancel to give an integer-valued index. Our derivation is based on axial current non-conservation — an exact operator identity valid on any four-manifold — and on the existence of a center symmetric, or approximately center symmetric, boundary holonomy (Wilson line). We expect the index theorem to usefully apply to many physical systems of interest, such as low temperature (large S^1, confined) phases of gauge theories, center stabilized Yang-Mills theories with vector-like or chiral matter (at S^1 of any size), and supersymmetric gauge theories with supersymmetry-preserving boundary conditions (also at any S^1). In QCD-like and chiral gauge theories, the index theorem should shed light into the nature of topological excitations responsible for chiral symmetry breaking and the generation of mass gap in the gauge sector. We also show that imposing chirally-twisted boundary condition in gauge theories with fermions induces a Chern-Simons term in the infrared. This suggests that some QCD-like gauge theories should possess components with a topological Chern-Simons phase in the small S^1 regime.

Keywords: Solitons Monopoles and Instantons, Nonperturbative Effects, Chern-Simons Theories, Anomalies in Field and String Theories

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1 Introduction

1.1 Motivation

A method to study non-perturbative aspects of an asymptotically free gauge theory on R^4 is to begin with a compactification on $R^3 \times S^1$. S^1 may be either a spatial or temporal circle, determined according to the spin connection of fermions. In asymptotically free gauge theories, the size of S^1 is a control parameter for the strength of the running coupling at the scale of compactification. At radius much smaller than the strong length scale, the theory is weakly coupled and at large radius, it is strongly coupled. It is well-known that certain aspects of weakly coupled gauge theories are amenable to perturbative treatment. It is less known that such gauge theories also admit a semi-classical non-perturbative treatment if the boundary Wilson line (Polyakov loop) satisfies certain conditions.

Let $\mathrm{U}(x)$ denote the holonomy of Wilson line wrapping the S^1 circle:

$$\mathrm{U}(x) = P \exp i \oint A_4 dy\,, \qquad x \in R^3, \;\; y \in S^1\,, \tag{1.1}$$

where $A_4(x,y)$ is the component of the gauge field along the compactified direction. If the eigenvalues of the A_4 field (which are gauge invariant) repel each other in the weak coupling regime, for a dynamical reason or due to a deformation described below, then the boundary value of the A_4 field as $|x| \to \infty$ takes the form:

$$A_4\big|_\infty = \mathrm{diag}(\hat{v}_1, \hat{v}_2, \ldots \hat{v}_N)\,, \qquad \hat{v}_1 < \hat{v}_2 < \ldots < \hat{v}_N\;. \tag{1.2}$$

In the weak coupling regime, A_4 behaves as a compact adjoint Higgs field and the vacuum configuration (1.2) induces gauge symmetry breaking, $G \to \mathbf{Ab}(G)$, to the maximal abelian subgroup $\mathbf{Ab}(G)$. This means that there exist a plethora of stable topological excitations in such four dimensional gauge theories, such as magnetic monopoles, magnetic bions, instantons and other interesting (stable) composites.

It is well-known that in a thermal set-up at sufficiently high temperatures (and weak coupling), dynamics disfavors configurations for Wilson lines such as (1.2) [1], and favors configurations for which $A_4\big|_\infty = \mathrm{diag}(\hat{v}, \hat{v}, \ldots \hat{v}) \;\; = (0, 0, \ldots, 0)$. In such cases, the effect of topological excitations is suppressed by volume factors, and semi-classical techniques do not usefully apply [1]. Due to these legitimate reasons, semi-classical methods in finite temperature setting have not received wide attention so far (although see [2]), and are not part of the common-place techniques to study Yang-Mills theory and non-supersymmetric Yang-Mills theory with vector-like and chiral matter, compactified on a circle.

However, there are at least three ways to make such boundary values of Wilson lines stable at weak coupling. These are: *a.*) center-stabilizing double-trace deformations, *b.*) adjoint fermions with periodic boundary conditions, or mixed representations of adjoints and a few complex representation fermions all with periodic boundary conditions, and *c.*) supersymmetry and supersymmetry preserving boundary conditions. In this sense, the case of non-trivial holonomy (1.2) at weak coupling is as generic as the high-temperature trivial holonomy. In particular, with the center stabilizing double-trace deformations, certain gauge theories on R^4, such as Yang-Mills theory and vector-like and even chiral theories can be smoothly connected to small $S^1 \times R^3$ [3–5]. There already exist evidence from lattice gauge theory (where deformations were also suggested independently to explore phases of partial center symmetry breaking) that the conjecture of smoothness holds for Yang-Mills theory [6].[1] Therefore, there is currently a strong incentive to study in detail the topological excitations on $R^3 \times S^1$ and the index theorems associated with these excitations.

Our interest is in the index of the Dirac (or Dirac-Weyl) operator $\hat{D}$ (1.5) in the background of topological excitations pertinent to the gauge theory on $R^3 \times S^1$. The reason that this is interesting for non-perturbative physics is two-fold:

[1] The center-stabilized small-S^1 regime of gauge theories is amenable to both numerical lattice simulations and non-perturbative semi-classical techniques. In this sense, this regime provides a first example in which we can confront a controlled approximation, including non-perturbative effects, with the lattice, and is, in our opinion, an important opportunity for both lattice and continuum gauge field theory.

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- The topological excitations with non-vanishing index will carry compulsory fermion zero modes attached to them, and may induce chiral symmetry breaking. Generically, the fermionic index of a monopole operator on $R^3 \times S^1$ is (much) smaller than the APS index for the BPST instanton. Thus, they are in principle more relevant for low energy phenomena.

- The generation of mass gap (and confinement) for gauge fluctuations, in the weak coupling regime, requires the existence of topological excitations with vanishing index. In typical QCD-like and chiral gauge theories, most of the leading topological excitations (monopoles) carry fermionic zero modes, hence cannot contribute to the mass gap. Therefore, the index theorem can be used to identify composite topological excitations (such as magnetic bions) for which the sum of individual indices add up to zero.

A simple example which illustrates both issues is Yang-Mills theory with adjoint fermions (QCD(adj)) and $\mathcal{N} = 1$ SYM. In both cases, magnetic monopole operators (which appear at order $e^{-S_0} = e^{-8\pi^2/(g^2N)}$ in the semi-classical e^{-S_0} expansion) induce a certain chiral condensate. However, the topological excitations responsible for the existence of mass gap of the dual photon and thus for confinement are the magnetic bions with vanishing index, which appear at order e^{-2S_0}. The index theorem on $R^3 \times S^1$ should help us identify both classes of non-perturbative topological excitations for any gauge theory.

The non-perturbative semi-classical analysis provides reliable information about the gauge theory in the weak coupling regime. However, the semi-classical treatment does not extend over to the large radius, strong coupling regime, where the eigenvalues of A_4 fluctuate rapidly, and there is no "Higgs regime" where the long-distance theory abelianizes. In the partition function, we need to sum over all gauge inequivalent configurations. At $x = \infty$, A_4 field can acquire a profile consistent with the unbroken center symmetry, such as (1.2). In fact, the boundary Wilson line (1.2) defines an isotropy group at infinity [1]: $G_{A_4|_\infty} = \{g \in G \mid g\mathrm{U}(\infty)g^\dagger = \mathrm{U}(\infty)\}$. For example, in low temperature pure Yang-Mills theory, the isometry group is isomorphic to the maximal abelian subgroup, $G_{A_4|_\infty} \sim \mathbf{Ab}(G)$. This does not mean that a dynamical abelianization takes place in this regime, nor semi-classical techniques apply. However, the index theorem for the Dirac operator can be interpolated from R^3 to R^4. The index theorem is valid at any radius, regardless of the value of the coupling constant.

Although the index theorem and topological excitations consistent with the isotropy group $G_{A_4|_\infty}$ continue to exist in the large S^1 strong-coupling domain, the semi-classical techniques no longer usefully apply. Nonetheless, we believe that there is value in studying the form of the topological operators, dictated by the appropriate index theorem, and at least qualitatively study their dynamical effects. This is the goal of the recent "deformation program".

1.2 Outline

We introduce our notation in section 1.3 below. In section 2, we begin the calculation of the index. Our calculation can be thought of as a generalization of that of [8, 9], see also [10]. We show that the index on $R^3 \times S^1$ has two contributions — a topological charge and surface term contribution.

In section 2.1, we first calculate the index for static monopole backgrounds. The surface term contribution, section 2.1.1, is expressed in terms of the η-invariant of the boundary Dirac operator, while the topological charge contribution is given in section 2.1.2. The final formula for the index in the "static" background is eq. (2.32) for the fundamental of SU(N) and eqs. (B.1), (B.4) from appendix B for other representations. The calculation of the index in a Kaluza-Klein ("winding") monopole background is given in section 2.2, with the result for the fundamental of SU(N) in (2.44), and in (B.7) for general representations.

We note that an expression for the index on $R^3 \times S^1$ similar to ours — given in terms of the topological charge and the η-invariant — can be extracted from the appendix of ref. [13]. The contribution of this paper consists of: *a.*) a derivation of the index accessible to physicists along the lines given in the physics literature for R^3 and by using exact operator identities valid on any four-manifold and *b.*) a calculation of the index in specific backgrounds and a discussion of its jumps — properties which are of interest for concrete quantum field theory applications.

In section 3, we discuss in some more detail the index for the three lowest representations of SU(2) and the fundamental of SU(N). We explain the jumps of the index which occur as the ratio of boundary holonomy to the size of S^1 is varied.

In section 4, we explain the relation to the Callias [11] and APS [12] indices.

In section 5, we consider the generation of fermion-loop induced Chern-Simons terms on $R^3 \times S^1$. We show when Chern-Simons terms are induced and how their coefficients are quantized. We consider the effect of turning on of discrete Wilson lines for background fields gauging anomalous flavor symmetries (similar effects are known from the string literature). The resulting Chern-Simons terms have a profound effect on the phase structure of the theory on $R^3 \times S^1$.

Finally, appendix A contains another calculation of the η-invariant. In appendix B, we give formulae for the index for general representations.

1.3 Notation

We take the four-dimensional Euclidean Dirac operator of a vector-like fermion in the representation $\mathcal{R}$ to be:

$$\hat{D} \equiv \gamma_\mu D_\mu, \;\; D_\mu \equiv \partial_\mu + i A_\mu^a T^a \; . \tag{1.3}$$

We use hermitean T^a's, obeying Tr $T^a T^b = T(\mathcal{R})\delta^{ab}$, taking T(fund.)= 1/2 for SU(N). To further set and check notation, note that we use, in a given representation, $\psi \to U\psi$, $A_\mu \to U A_\mu U^\dagger - iU\partial_\mu U^\dagger$ under gauge transformations, hence $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$. Roman indices run from $1 \ldots 3$, while Greek indices span $1 \ldots 4$; the $x^4 \equiv y$ direction is periodic, $y \equiv y + L$. The hermitean Euclidean γ-matrix basis we use is:

$$\gamma_k = \sigma_1 \otimes \sigma_k \, , \quad \gamma_4 = -\sigma_2 \otimes \sigma_0 \, , \quad \gamma_5 = \sigma_3 \otimes \sigma_0 \, , \tag{1.4}$$

where σ_k are Pauli matrices and σ_0 is the unit matrix. The vector-like Dirac operator (1.3) is:

$$\hat{D} = \begin{pmatrix} 0 & \sigma_k D_k + i\sigma_0 D_4 \\ \sigma_k D_k - i\sigma_0 D_4 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & -D^\dagger \\ D & 0 \end{pmatrix} \, . \tag{1.5}$$

In the above equation, we defined the 2×2 Weyl operator D, obeying:

$$D^\dagger D = -D_\mu D_\mu + \sigma^m \frac{1}{2}\epsilon_{mkl} F_{kl} + \sigma_k F_{4k} = -D_\mu D_\mu + 2\sigma^m B^m \tag{1.6}$$

$$D D^\dagger = -D_\mu D_\mu + \sigma^m \frac{1}{2}\epsilon_{mkl} F_{kl} - \sigma_k F_{4k} = -D_\mu D_\mu \,. \tag{1.7}$$

Here, D_μ is as defined in (1.3), and we assumed, without loss of generality, that the background of interest is anti self-dual, namely that $F_{4k} = \frac{1}{2}\epsilon_{kpq}F_{pq} \equiv B_k$ (all expressions can be easily generalized for self-dual backgrounds). In this paper, we will use "Tr" to denote traces of operators over spacetime as well as spinor indices, while "tr" will refer to traces over spinor indices only.

We are interested in computing the index of the Dirac operator in topologically nontrivial backgrounds on $R^3 \times S^1$, generalizing the R^3 result of [11]. The simplest example of a nontrivial background is given by the three-dimensional SU(2) Prasad-Sommerfield (PS) solution of unit magnetic charge, embedded in $R^3 \times S^1$. The other backgrounds on interest can be constructed by taking superpositions of the fundamental monopoles and other solutions, obtained by non-periodic "gauge" transformations.

The PS solution is "static" (i.e. y-independent) and the A_4-component of the gauge field plays the role of the Higgs field. For example, consider the anti self-dual solution, which obeys $F_{4k} = -\frac{1}{2}\epsilon_{4kpq}F_{pq} = \frac{1}{2}\epsilon_{kpq}F_{pq} \equiv B_k$. In our conventions and in regular ("hedgehog") gauge, the SU(2) solution reads:

$$A_4 = A_4^a(r,v)\, T^a = \hat{r}^a f(r,v)\, T^a \,, \quad A_j = A_j^a(r,v)\, T^a = \epsilon_{jba}\hat{r}^b g(r,v)\, T^a \,, \tag{1.8}$$

where $\hat{r}^a = \frac{r^a}{r}$ and:

$$f(r,v) = \frac{1}{r} - v \coth vr \,, \quad g(r,v) = -\frac{1}{r} + \frac{v}{\sinh vr} \,. \tag{1.9}$$

The asymptotics of the B_k, A_4 fields of the PS solution at infinity are:

$$\begin{aligned} A_4\big|_\infty &= -v\, \hat{r}^a T^a \left(1 - \frac{1}{vr} + \ldots\right) \\ B_k\big|_\infty &= \frac{\hat{r}^k}{r^2}\, \hat{r}^a T^a + \ldots \,, \end{aligned} \tag{1.10}$$

where dots denote terms that vanish as e^{-vr}. To cast the solution in string gauge, we need to gauge transform $\hat{r}^a T^a \to T^3$. The asymptotics of A_4 and B_k in string gauge are obtained from (1.10) by replacing $\hat{r}^a T^a$ with T^3, for example the asymptotics of SU(2)-holonomy is $A_4|_\infty = \frac{1}{2}\mathrm{diag}(-v, v)$.

For the applications we have in mind, we want to also consider monopole solutions of SU(N). The SU(2) PS solution considered above can be embedded in SU(N) as described in [9]. For simplicity, we will use the fundamental generators of SU(N) to describe the embedding (a description of the embedding using roots and weights can be also given, see [9]; however, we find that for our purposes using $N \times N$ matrices is both sufficient and illuminating). The general form of the asymptotics of the Higgs field A_4 is:

$$\begin{aligned} A_4\big|_\infty &= \mathrm{diag}(\hat{v}_1, \hat{v}_2, \ldots \hat{v}_N) \,, \\ &\hat{v}_1 < \hat{v}_2 < \ldots < \hat{v}_N, \qquad \sum_{i=1}^N \hat{v}_i = 0 \end{aligned} \tag{1.11}$$

where, without loss of generality, we have ordered the eigenvalues as in our SU(2) example (for example, eq. (1.10) corresponds to taking $\hat{v}_1 = -\hat{v}_2 = -\frac{v}{2}$). A background with an additional overall U(1) "Wilson line" a_0, often also called "real mass" term (when fermions are included), allows the holonomies $\hat{v}_j$ to be more general:

$$\hat{v}_j \to \hat{v}_j + \frac{1}{\sqrt{2N}}\, a_0, \tag{1.12}$$

where we also normalized the overall U(1) generator multiplying a_0 to Tr $T^2 = 1/2$. Including a non-vanishing a_0 can be used to incorporate different boundary conditions for the fermions in all our formulae.

The asymptotic form of the U(N) holonomy (1.11), (1.12) admits N types of elementary monopoles; N-1 of these are associated with the positive simple roots α_i of the SU(N) Lie algebra, for which:

$$\alpha_i \cdot \mathbf{H} = \frac{1}{2}\mathrm{diag}(0,\ldots,\underbrace{1}_{i},\underbrace{-1}_{i+1},\ldots,0), \qquad i = 1, \ldots N-1\,, \tag{1.13}$$

where $\mathbf{H} = (H^1, \ldots, H^{N-1})$, where H^a denote the Cartan generators of SU(N). The Cartan generators and simple roots are normalized as Tr $H^a H^b = \frac{1}{2}\delta^{ab}$, $\alpha_i \cdot \alpha_j = \delta_{i,j} - \frac{1}{2}\delta_{i,j\pm 1}$. The $N^{\rm th}$ type of fundamental monopole arises due to compactness of the "Higgs" field A_4, and is associated with the "affine" root:

$$\alpha_N \cdot \mathbf{H} \equiv -\sum_{j=1}^{N-1} \alpha_j \cdot \mathbf{H} \;= \frac{1}{2}\mathrm{diag}(-1, 0, 0, \ldots, 1)\,. \tag{1.14}$$

A monopole solution corresponding to the $i^{\rm th}$ simple root (1.13) of SU(N) can be constructed from (1.10) as follows. First, rewrite the holonomy (1.11):

$$A_4\big|_\infty = \mathrm{diag}(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \underbrace{V}_{i}, \underbrace{V}_{i+1}, \hat{v}_{i+2}, \ldots, \hat{v}_N) + \frac{1}{2}\mathrm{diag}(0, 0, \ldots, 0, \underbrace{-\tilde{V}}_{i}, \underbrace{\tilde{V}}_{i+1}, 0, \ldots, 0)\,, \tag{1.15}$$

where $V = \frac{1}{2}(\hat{v}_{i+1} + \hat{v}_i)$ and $\tilde{V} = \hat{v}_{i+1} - \hat{v}_i$. We now diagonally embed the SU(2) generators τ^a into SU(N), such that their only nonzero elements are equal to one-half the Pauli matrices embedded in a 2×2 square along the diagonal of the $N \times N$ matrices (thus, their diagonal elements are the $i^{\rm th}$ and $i{+}1^{\rm th}$ ones singled out in (1.15)). With this embedding it is easy to explicitly verify that:

$$\begin{aligned} A_4 &= \hat{r}^a f(r, \tilde{V})\tau^a + \mathrm{diag}(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_{i-1}, \underbrace{V}_{i}, \underbrace{V}_{i+1}, \hat{v}_{i+2}, \ldots, \hat{v}_N)\,, \\ A_m &= \epsilon_{mba}\hat{r}^b g(r, \tilde{V})\tau^a\,, \end{aligned} \tag{1.16}$$

with $f(r, \tilde{V})$ and $g(r, \tilde{V})$ defined in (1.9) solves the anti-self-duality condition $F_{4k} = B_k$ inside SU(N). The large-radius asymptotics can be immediately read off (1.10) by replacing T^a with τ^a and inserting in (1.16).

Finally, a collection of $n_1, n_2, \ldots, n_{N-1}$ fundamental monopoles of the type corresponding to the 1^{st}, 2^{nd}, $\ldots$, $N-1^{\text{th}}$, respectively, simple root of SU(N), has an asymptotic magnetic field which is the natural generalization of (1.10) and is given, in the string gauge, by:

$$\begin{aligned} B^m\big|_\infty &= \frac{\hat{x}^m}{|x|^2} \sum_{i=1}^{N-1} n_i \left(\alpha_i \cdot \mathbf{H}\right) \\ &= \frac{1}{2} \frac{\hat{x}^m}{|x|^2} \, \mathrm{diag}(n_1, n_2 - n_1, \ldots, n_j - n_{j-1}, \ldots, -n_{N-1}) \ . \end{aligned} \tag{1.17}$$

The asymptotic form of the SU(N) holonomy is as in (1.11) and the string gauge asymptotics of the gauge field is best described in polar coordinates, with A_ϕ its only nonvanishing component:

$$A_\phi\big|_\infty = \frac{1-\cos\theta}{2} \, \mathrm{diag}(n_1, n_2 - n_1, \ldots, n_j - n_{j-1}, \ldots, -n_{N-1}) \ . \tag{1.18}$$

The Kaluza-Klein monopole solution corresponding to the affine root (1.14) will be constructed in section 2.2.

2 The index for Dirac operator on $R^3 \times S^1$

We define the Callias index of a Weyl fermion (with equation of motion $D\psi = 0$ and D defined in (1.5)) in the representation $\mathcal{R}$ on $S^1 \times R^3$ as in [8, 11]:

$$I_{\mathcal{R}} = \lim_{M^2 \to 0} \mathrm{Tr} \, \frac{M^2}{D^\dagger D + M^2} - \mathrm{Tr} \, \frac{M^2}{D D^\dagger + M^2} \ . \tag{2.1}$$

This is the definition most convenient for explicit calculations, despite the fact that in the locally four dimensional case of interest an additional regularization will be required, having to do with the need to perform the sum over the Kaluza-Klein tower implicit in (2.1). Nonzero discrete eigenvalues do not contribute to the formal expression (2.1) — if ψ is an eigenfunction of $D^\dagger D$ with a nonzero eigenvalue, $D\psi$ is an eigenfunction of $DD^\dagger$ with the same eigenvalue and so their contributions to the trace cancel — hence $I_{\mathcal{R}}$ counts the number of zero modes of D minus the number of zero modes of $D^\dagger$; the continuous spectrum also does not contribute to (2.1) if all $\hat{v}_j$ are different, see the discussion in the appendix of ref. [8]. The arguments given there continue to hold on $R^3 \times S^1$ and we will not repeat them here — as will become clear from our results, $I_{\mathcal{R}}$ of eq. (2.1) always yields an integer value for finite action backgrounds on $R^3 \times S^1$. Furthermore, we will show that the index reduces to the Callias index in the appropriate limit and experiences discontinuous jumps, which can also be explained physically, upon changing the ratio of the circumference of S^1 (L) to the holonomies at infinity ($\hat{v}_j$) .

Using the operator $\hat{D}$ from (1.5) and our notation for γ_5 (1.4), we find that:

$$I_{\mathcal{R}}(M^2) = \mathrm{Tr} \, \gamma_5 \frac{M^2}{-\hat{D}^2 + M^2} = M \mathrm{Tr} \, \gamma_5 \frac{\hat{D} + M}{-\hat{D}^2 + M^2} \, , \tag{2.2}$$

where the second identity is true because of cyclicity of trace and $\gamma_5 \hat{D} = -\hat{D}\gamma_5$. Finally we can cancel the $\hat{D} + M$ factor between numerator and denominator and arrive at the expression we will actually use:

$$I_{\mathcal{R}}(M^2) = M \mathrm{Tr}\; \gamma_5 \frac{1}{-\hat{D} + M} \; . \tag{2.3}$$

In our study, we closely follow the derivation of the Callias index on R^3 of ref. [8], paying respect to the differences due to the locally four-dimensional nature of spacetime. The main difference — apart from the already mentioned sum over Kaluza-Klein modes — occurs in the very first step below and has to do with the fact that anomalies occur in a locally four dimensional spacetime. To elucidate, we note that:

$$\langle x \left| \frac{1}{\hat{D} - M} \right| y \rangle = \langle \psi(x) \overline{\psi}(y) \rangle \,, \tag{2.4}$$

where $\langle \ldots \rangle$ denotes an expectation value in a Euclidean quantum field theory of a Dirac fermion $\psi, \overline{\psi}$ with action $-S = \overline{\psi}(-\hat{D}+M)\psi$. For such theories in a locally four dimensional background the following operator identity holds:

$$\partial_\mu J^5_\mu \equiv \partial_\mu (\overline{\psi} \gamma_\mu \gamma_5 \psi) = -2M \overline{\psi} \gamma_5 \psi - \frac{T(\mathcal{R})}{8\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} \; . \tag{2.5}$$

The index (2.3), via (2.4), (2.5), can be rewritten as:

$$\begin{aligned} I_{\mathcal{R}}(M^2) &= -M \;\mathrm{Tr}\; \gamma_5 \langle \psi \overline{\psi} \rangle = M \int d^3x \int_0^L dy \; \langle \overline{\psi} \gamma_5 \psi \rangle \\ &= -\frac{1}{2} \int_{S^2_\infty} d^2\sigma^k \int_0^L dy \; \langle J^5_k \rangle - \frac{T(\mathcal{R})}{16\pi^2} \int d^3x \int_0^L dy \; G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} \,, \end{aligned} \tag{2.6}$$

where we used periodicity of the current on S^1 to argue that the integral of $\partial_y \langle J^5_4 \rangle$ vanishes. Eq. (2.6) is our main tool, allowing us to smoothly interpolate the index from R^3 to R^4 by varying the size of the circle and the appropriate background. As a first simple check, take the limit of an infinite L, i.e. R^4, where eq. (2.6) becomes:

$$I_{\mathcal{R}}(M^2) = -\frac{1}{2} \int_{S^3_\infty} d^3\sigma^\mu \; \langle J^5_\mu \rangle - \frac{T(\mathcal{R})}{16\pi^2} \int d^4x \; G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} \; . \tag{2.7}$$

This is the index theorem[2] appropriate for a BPST instanton background (provided that the surface term vanishes: that this is so follows from the fact that the surface contribution in (2.7) could only be due to BPST fermion zero modes, as the nonzero modes vanish exponentially at S^3_∞ and so does their current; the fermion zero modes in an instanton fall off as a powerlaw $\psi_0|_{x \to \infty} \sim \frac{\rho}{|x|^3}$, in nonsingular gauge [14]).

[2] To avoid (or add) confusion, recall that the index (2.1) for a fundamental Weyl fermion in an anti-selfdual instanton should be $+1$, as it is D, in the notation of section 1.3, that has a normalizable zero mode.

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Going back to $R^3 \times S^1$, consider now the integral over $S^2_\infty \times S^1$ in (2.6). We rewrite the surface term in (2.6) as follows:

$$I^1_{\mathcal{R}}(M^2) \equiv -\frac{1}{2}\int\limits_{S^2_\infty} d^2\sigma^k \int\limits_0^L dy\, \langle J^5_k\rangle = -\frac{1}{2}\int\limits_{S^2_\infty} d^2\sigma^k \int\limits_0^L dy\, \mathrm{tr}\langle x \left|\gamma^k\gamma_5 \frac{1}{-\hat{D}+M}\right| x\rangle \quad (2.8)$$

$$= -\frac{1}{2}\int\limits_{S^2_\infty} d^2\sigma^k \int\limits_0^L dy\, \mathrm{tr}\langle x \left|\left(\gamma^k\gamma_5\hat{D}\frac{1}{-\hat{D}^2+M^2}\right)\right| x\rangle\,, \quad (2.9)$$

where we performed the operations that led to eq. (2.3) in reverse. Further, from (1.5), the expressions (1.6) for $D^\dagger D$ and $DD^\dagger$, and the explicit form (1.4) of the γ-matrices, we have:

$$I^1_{\mathcal{R}}(M^2) = \frac{1}{2}\int\limits_{S^2_\infty} d^2\sigma^k \int\limits_0^L dy \mathrm{tr}\langle x|\ \sigma^k\sigma^l D_l \left(\frac{1}{-D^2_\nu + M^2 + 2\sigma^m B^m} - \frac{1}{-D^2_\nu + M^2}\right)|x\rangle$$

$$-\frac{1}{2}\int\limits_{S^2_\infty} d^2\sigma^k \int\limits_0^L dy \mathrm{tr}\langle x|\ i\sigma^k D_4 \left(\frac{1}{-D^2_\nu + M^2 + 2\sigma^m B^m} + \frac{1}{-D^2_\nu + M^2}\right)|x\rangle\,, \quad (2.10)$$

and we recall that (2.10) is written for an anti self-dual background. The final formula for the index which will be used in our further computations is:

$$I_{\mathcal{R}}(M^2) = I^1_{\mathcal{R}}(M^2) - \frac{T(\mathcal{R})}{16\pi^2}\int d^4x\, G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} \equiv I^1_{\mathcal{R}} + I^2_{\mathcal{R}}\,, \quad (2.11)$$

with $I^1_{\mathcal{R}}$ defined in (2.10) and $I^2_{\mathcal{R}}$, the topological charge contribution to the index, in (2.11).

2.1 The index in a "static" BPS monopole background and Callias index

Consider first a 3d BPS "static" monopole background, independent on the S^1 coordinate. Physical intuition tells us that if we consider a small S^1, hence weak coupling, we expect the index on $R^3 \times S^1$ to be the same as that on R^3 provided $Lv \ll 1$, such that KK modes do not influence physics at scales of order the size of the monopole. On the other hand, one expects that when $Lv \gg 1$, the index can differ from the one on R^3. To study how this expectation plays out in detail and under what conditions the index can jump, in the following sections we successively evaluate the two contributions to the index (2.11).

2.1.1 Surface term contribution

To evaluate the contribution of the surface term (2.10), we note that at infinity the dominant terms in the expansion of the operators appearing in $I^1_R(M^2)$ in the static BPS background are:

$$-D^2_\nu + M^2 \simeq -\partial^2_m + M^2 - D^2_4,\ \text{with}\ -iD_4 \to \frac{2\pi n}{L} + A_4\,, \quad (2.12)$$

where we used the string-gauge asymptotics of A_4 (1.11) and A_m (1.18). We now expand the surface term contribution, recalling that $B^m \sim r^{-2}$, observing that only the second term in (2.10) contributes after the Pauli matrix traces are taken, and using (2.12):

$$I^1_{\mathcal{R}}(M^2) = 2\int_0^L dy \int_{S^2_\infty} d^2\sigma^k \text{ tr } \langle x;y \left| iD_4 \frac{1}{-\partial_m^2 + M^2 - D_4^2} B^k \frac{1}{-\partial_m^2 + M^2 - D_4^2} \right| x;y \rangle \ . \tag{2.13}$$

Next, we substitute the asymptotic form for a "static" BPS solution, eq. (1.17), to obtain[3] using (2.12) to replace iD_4:

$$I^1_{\mathcal{R}}(M^2) = -\int_{S^2_\infty} d^2\sigma^k \frac{\hat{x}_k}{|x|^2} \sum_{p=-\infty}^{\infty} \sum_{j=1}^{N} \left(\hat{v}_j + \frac{2\pi p}{L}\right)(n_j - n_{j-1}) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\left[k^2 + M^2 + (\hat{v}_j + \frac{2\pi p}{L})^2\right]^2} \ . \tag{2.14}$$

After taking the three dimensional momentum and surface integrals ($d^2\sigma^k \equiv |x|^2 \hat{x}_k \, d\Omega_{S^2}$), as well as the $M^2 \to 0$ limit, the surface contribution to the index becomes:

$$I^1_{\mathcal{R}}(0) = -\frac{1}{2} \sum_{j=1}^{N} (n_j - n_{j-1}) \sum_{p=-\infty}^{\infty} \frac{\hat{v}_j + \frac{2\pi p}{L}}{|\hat{v}_j + \frac{2\pi p}{L}|} \ . \tag{2.15}$$

The Kaluza-Klein (KK) mode sum in (2.15) is a periodic generalization of the sign function, which appears in the Callias index for gauge theories on R^3 (upon taking $L \to 0$ only the $p = 0$ term contributes in the sum and so (2.15) reproduces the Callias index result, see appendix B). Such a generalization is necessary, since on $R^3 \times S^1$ the eigenvalues of the "Higgs" field A_4 are compact and the index should be a periodic function of the expectation values of A_4, with periodicity determined by the representation $\mathcal{R}$.

The KK sum (2.15) can also be thought of as a sum of the indices of a KK tower of three-dimensional Dirac operators, each that of a KK fermion of mass $\frac{2\pi p}{L}$. The Callias index theorem shows that for a given Higgs vev only a finite number of massive operators in the KK tower have a nonvanishing index (essentially, those with $|m| < \mathcal{O}(|v|)$), thus only a few terms in the sum over indices of KK Dirac operators can contribute to the index. While following this logic is a quick way to find our formula for the index for static backgrounds, recall that there is also a non-integer topological charge contribution given by the second term in (2.10), which should be cancelled by a corresponding non-integer contribution to (2.15) to yield an integer value. Thus, to obtain a formula for the index that works for general backgrounds [1], specified by the holonomy at infinity, magnetic charge, and topological charge, we must regulate the sum over KK modes in (2.15).

For a given j, the KK sum is equal to $\eta_j[0]$, the spectral asymmetry of the differential operator $h_j = i\frac{d}{dy} + \hat{v}_j$ acting on the space of periodic functions $f(y) = f(y + L)$. The η-invariant is defined by analytic continuation from sufficiently large $\text{Re}(s) > 0$ of:

$$\eta[v_j, s] \equiv \eta_j[s] \equiv \sum_{\lambda \neq 0} \frac{\text{sign}\lambda}{|\lambda|^s}, \tag{2.16}$$

[3] Recall that $n_0 = n_N = 0$ is understood for the static solution.

where λ are the eigenvalues of h_j.[4] Thus the surface term contribution to the index is:

$$I^1_{\mathcal{R}}(0) = -\frac{1}{2}\sum_{j=1}^{N}(n_j - n_{j-1})\,\eta_j[0]\,. \tag{2.19}$$

To calculate $\eta_j[0]$, begin with its definition (2.16), rescaling both numerator and denominator by $\frac{2\pi}{L}$:

$$\eta_j[s] = \sum_{p=-\infty}^{\infty} \frac{\mathrm{sign}\left(\frac{\hat{v}_j L}{2\pi} + p\right)}{|\frac{\hat{v}_j L}{2\pi} + p|^s} = \sum_{p=-\infty}^{\infty} \frac{\mathrm{sign}\,(\hat{a}_j + p)}{|\hat{a}_j + p|^s}\,. \tag{2.20}$$

We defined:

$$\hat{a}_j \equiv \frac{\hat{v}_j L}{2\pi} - \left\lfloor \frac{\hat{v}_j L}{2\pi} \right\rfloor \subset (0,1)\,, \tag{2.21}$$

having noted that since η_j is a periodic function of $\hat{a}_j$ of unit period, by relabeling the KK modes, we can take the argument to lie in the fundamental interval $(0,1)$. Here $\lfloor x \rfloor$ is the floor function:

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \le x\}\,, \tag{2.22}$$

which denotes the largest integer smaller than x, and $\hat{x} = x - \lfloor x \rfloor$ is the fractional part of x.

It then follows that all terms in the sum (2.20) with $p \ge 0$ are positive, while the ones with $p < 0$ are negative, allowing us to write:

$$\eta_j[s] = \sum_{p\ge 0} \frac{1}{(\hat{a}_j + p)^s} - \sum_{p \ge 0} \frac{1}{(p + 1 - \hat{a}_j)^s} = \zeta(s, \hat{a}_j) - \zeta(s, 1 - \hat{a}_j)\,, \tag{2.23}$$

where $\zeta(s,x)$ is the incomplete zeta-function. Finally [15], since $\zeta(0,x) = \frac{1}{2} - x$, we find our final expression for $\eta_j[0]$:

$$\eta_j[0] = \frac{1}{2} - \hat{a}_j - \left(\frac{1}{2} - (1 - \hat{a}_j)\right) = 1 - 2\hat{a}_j = 1 - 2\frac{\hat{v}_j L}{2\pi} + 2\left\lfloor \frac{\hat{v}_j L}{2\pi} \right\rfloor\,. \tag{2.24}$$

For another calculation of the η-invariant, see appendix A.

From (2.24), the surface term contribution (2.15) to the index for the fundamental representation of SU(N) becomes:

$$I^1_{fund.}(0) = -\sum_{j=1}^{N}(n_j - n_{j-1})\left(\frac{1}{2} - \frac{\hat{v}_j L}{2\pi} + \left\lfloor \frac{\hat{v}_j L}{2\pi} \right\rfloor\right)\,. \tag{2.25}$$

[4] An equivalent way to to define the η-invariant is via its integral representation. Let $H = i\frac{d}{dy} + A_4$. Then,

$$\eta[H,s] \equiv \mathrm{tr}\frac{H}{(H^2)^{(s+1)/2}} \equiv \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty dt\; t^{(s-1)/2}\mathrm{tr}[He^{-H^2 t}]\,. \tag{2.17}$$

This representation makes sense for large $\mathrm{Re}(s) > 0$ and admits a holomorphic extension to the whole complex plane. This discussion is completely parallel to much often encountered ζ function regularization, for which:

$$\zeta[H,s] \equiv \mathrm{tr}[H^{-s}] \equiv \frac{1}{\Gamma(s)}\int_0^\infty dt\; t^{(s-1)}\mathrm{tr}[e^{-Ht}]\,. \tag{2.18}$$

2.1.2 Topological charge contribution

Consider now the second term in (2.11)—the topological charge contribution to the index, which is well-known to be a surface term:

$$I^2_{\mathcal{R}}(0) = -2T(\mathcal{R})Q = -\frac{T(\mathcal{R})}{16\pi^2}\int d^3x \int_0^L dy\, G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} = -\frac{T(\mathcal{R})}{16\pi^2}\int_0^L dy \int_{S^2_\infty} d^2\sigma^m K^m\,, \quad (2.26)$$

The topological current is:

$$K^\mu = 4\epsilon^{\mu\nu\lambda\kappa}\mathrm{tr}\left(A_\nu\partial_\lambda A_\kappa + \frac{2i}{3}\,A_\nu A_\lambda A_\kappa\right)\,. \quad (2.27)$$

In writing the surface integral in (2.26), we used the fact that for the static BPS background K^μ is a periodic function of y. To evaluate (2.26) we note that the spatial component of K^μ can be rewritten as:

$$K^m = 4\epsilon^{mij}\mathrm{tr}\left(A_4F_{ij} - A_i\partial_4 A_j - \partial_i(A_4A_j)\right)\,. \quad (2.28)$$

Now we use $\epsilon^{ijk}F_{jk} = 2B^i$ and the fact that in the static anti self-dual BPS background (1.8), (1.10), assuming SU(2) for now, $8\mathrm{tr}A_4B_m\big|_\infty = -8v\frac{\hat{r}^m}{r^2}\hat{r}^b\hat{r}^c\mathrm{tr}T^bT^c = -4v\frac{\hat{r}^m}{r^2}$. Thus, the only contribution to the surface integral (2.26) comes from the first term in K^m, yielding, for $T(\mathcal{R}) = 1/2$:

$$I^2_{fund.,\mathrm{SU}(2)}(0) = \frac{1}{32\pi^2}\,4\pi L\; 4v = \frac{Lv}{2\pi}\,. \quad (2.29)$$

This is, of course, the known result for the negative of the topological charge of an anti self-dual BPS monopole.

To obtain the SU(N) result in the multimonopole background, it is best to transform the surface integral (2.26) to string gauge and use (1.11), (1.17). The singular nature of the static gauge transformation does not change the periodicity of K_μ used in (2.26) and does not affect the surface integral.[5] Thus, for an arbitrary representation of SU(N) the topological charge contribution to the index is:

$$\begin{aligned} I^2_{\mathcal{R}}(0) &= -\frac{T(\mathcal{R})}{16\pi^2}\int_0^L dy \int_{S^2_\infty} d^2\sigma^m 8\mathrm{tr}[A_4B_m] \\ &= -2T(\mathcal{R})\sum_{j=1}^N (n_j - n_{j-1})\frac{L\hat{v}_j}{2\pi}\,. \end{aligned} \quad (2.30)$$

2.1.3 The final expression for the index

Combining the two contributions to the index, eqs. (2.30) and (2.25), gives our final formula for the index. Note that neither the topological charge contribution (2.30), nor the surface term (2.25) is an integer. However, in the combined result, the non-integer parts coming

[5] Note that, with $A^U_\mu = UA_\mu U^\dagger - iU\partial_\mu U^\dagger$, we have $K^\mu(A^U) = K^\mu(A) + \epsilon^{\mu\nu\lambda\kappa}\left(\frac{4}{3}\mathrm{tr}(U\partial_\nu U^\dagger\, U\partial_\lambda U^\dagger\, U\partial_\kappa U^\dagger) + 4i\,\partial_\nu\mathrm{tr}(A_\lambda U^\dagger\partial_\kappa U)\right)$ and the singular static gauge transformation does not introduce a shift to K_m.

from the two cancel neatly. With some work, our expression can also be extracted from the formulae in the appendix of [13]; it was derived here in a physicists' manner by using eq. (2.5), the axial-current non-conservation which is an exact operator identity valid on any 4-manifold. In this respect, our derivation is a natural generalization of [8].

For the fundamental representation of SU(N), adding (2.30) to (2.25), the index is:

$$\begin{aligned} I_{fund.}(n_1, n_2, \ldots, n_{N-1}) &= -\sum_{j=1}^{N}(n_j - n_{j-1})\left(\frac{1}{2} + \left\lfloor \frac{L\hat{v}_j}{2\pi} \right\rfloor\right) , \\ &= -\sum_{j=1}^{N-1} n_j \left(\left\lfloor \frac{L\hat{v}_j}{2\pi} \right\rfloor - \left\lfloor \frac{L\hat{v}_{j+1}}{2\pi} \right\rfloor\right) , \end{aligned} \tag{2.31}$$

where in the first line, as usual $n_N = n_0 = 0$.

It is fairly easy to extract the Callias index theorem from (2.31). Let us restrict $-\pi < L\hat{v}_j < \pi$ for all j. Then, $\frac{1}{2} + \left\lfloor \frac{L\hat{v}_j}{2\pi} \right\rfloor = \frac{1}{2}\text{sign }(\hat{v}_j)$ and (2.31) reduces to:

$$I_{fund.}(n_1, n_2, \ldots, n_{N-1}) = -\frac{1}{2}\sum_{j=1}^{N}(n_j - n_{j-1}) \text{ sign }(\hat{v}_j) \; = n_{j*}, \tag{2.32}$$

where $\hat{v}_{j*} < 0 < \hat{v}_{j*+1}$ and we used the the ordering of the holonomies' eigenvalues, eq. (2.32). In other words the fundamental representation fermion zero mode localizes at the $j^{*\text{th}}$ fundamental monopole, the known Callias index result.

2.2 The index in a "winding" BPS-KK monopole background

Another class of solutions that is crucial for describing the nonperturbative dynamics for nonzero L are the Kaluza-Klein monopoles, arising because of the compact nature of the "Higgs" field [16, 17]; see [18] for a semiclassical calculation elucidating their role in supersymmetric gluodynamics.

Let us recall the construction of the KK monopole solution corresponding to the "affine" root (1.14) of the SU(N) Lie algebra. We will construct the solution in analogy with the simple root monopoles given in section 1.3. To begin, note that we can rewrite the holonomy (1.11) as follows:

$$A_4 = -\tilde{V}\tau^3 + \text{diag}(V, \hat{v}_2, \ldots, , \ldots, \hat{v}_{N-1}, V)\,, \tag{2.33}$$

where now $V = \frac{1}{2}(\hat{v}_N + \hat{v}_1)$ and $\tilde{V} = \hat{v}_N - \hat{v}_1$. We take an SU(2) embedding in SU(N) via $\tau^{1,2,3}$ as:

$$\begin{aligned} (\tau^1)_{ij} &= \frac{1}{2}(\delta_{i1}\delta_{jN} + \delta_{iN}\delta_{j1}), & (\tau^2)_{ij} &= \frac{1}{2}(-i\delta_{i1}\delta_{jN} + i\delta_{iN}\delta_{j1}), \\ (\tau^3)_{ij} &= \frac{1}{2}(\delta_{i1}\delta_{j1} - \delta_{iN}\delta_{jN}), & i,j &= 1, \ldots, N. \end{aligned} \tag{2.34}$$

Clearly, the static self-dual monopole solution is, in complete analogy with the simple-root solutions (1.16):

$$\begin{aligned} A_4 &= \hat{r}^a f(r, \tilde{V})\tau^a + \text{diag}(V, \hat{v}_2, \ldots, \hat{v}_{N-1}, V)\,, \\ A_m &= \epsilon_{mba}\hat{r}^b g(r, \tilde{V})\tau^a \;. \end{aligned} \tag{2.35}$$

In the class of static solutions (2.35) is not a fundamental monopole but can be thought as a composite of the fundamental solutions based on simple roots. However, in theories with compact Higgs fields it can be used to construct the Kaluza-Klein monopole. To begin, note that the non-periodic "gauge transformation,"[6] defined via our fundamental SU(2) generator τ^3 embedded in SU(N) as described above:

$$\begin{aligned} U_1(y) &= e^{-i\frac{2\pi y}{L}\tau^3} = \mathrm{diag}(e^{-i\frac{\pi y}{L}}, 1, \dots, 1, e^{i\frac{\pi y}{L}}) \\ U_1(y+L) &= \mathrm{diag}(-1, 1, \dots, 1, -1)\; U_1(y) \end{aligned} \tag{2.36}$$

transforms periodic adjoint fields into periodic fields. At the same time, the asymptotic value of A_4 is shifted by $U_1(y)$:

$$A_4^{U_1} = A_4 + \frac{2\pi}{L}\tau^3 = -\left(\tilde{V} - \frac{2\pi}{L}\right)\tau^3 + \mathrm{diag}(V, \hat{v}_2, \dots \hat{v}_{N-1}, V)\ . \tag{2.37}$$

To construct the affine KK monopole, one starts with the static monopole solution (2.35) in a vacuum (2.33) with $\tilde{V}$ replaced by $\tilde{V}' = \frac{2\pi}{L} - \tilde{V}$. Denote by $A_\mu(\tilde{V}')$, $\mu = (4, m)$, the just described solution (2.35) in a vacuum given by $\tilde{V} \to \tilde{V}'$. Then one defines the field configuration:

$$A_\mu^{KK}(\tilde{V}) = U_2\left(A_\mu(\tilde{V}')\right)^{U_1} U_2^\dagger = U_2 U_1\left(A_\mu(\tilde{V}') - i\partial_\mu\right) U_1^\dagger U_2^\dagger\ . \tag{2.38}$$

Here U_2 is essentially the unit matrix except for its 11, NN, $1N$, and $N1$, elements, explicitly:

$$U_2 \equiv \begin{pmatrix} 0 & & & & 1 \\ & 1 & & & \\ & & 1 & & \\ & & \dots & & \\ & & & 1 & \\ -1 & & & & 0 \end{pmatrix}\ . \tag{2.39}$$

The point of (2.38) is that transforming $A_\mu(\tilde{V}')$ with U_1 leads to a twisted (i.e. y-dependent) solution in the vacuum with asymptotics given by (2.33) with $\tilde{V}$ replaced by $\tilde{V}' - \frac{2\pi}{L} = -\tilde{V}$. The role of the U_2 transformation acting on A_4 is to flip the sign of $\tilde{V}$ and thus generate a solution in the desired vacuum (2.33). The A_4 asymptotics of the KK monopole solution A_μ^{KK} is thus the desired (2.33), while the B-field flips sign at infinity due to the U_2 conjugation. Thus the KK monopole solution has magnetic charge opposite that of the corresponding anti self dual solution — its magnetic charge given by the affine root (1.14) and asymptotics (for n_N copies of the solution):

$$\begin{aligned} B^m_{KK}\big|_\infty &= -n_N \frac{\hat{x}^m}{|x|^2}\sum_{i=1}^{N}(\alpha_i \cdot \mathbf{H}) \\ &= \frac{n_N}{2}\frac{\hat{x}^m}{|x|^2}\mathrm{diag}(-1, 0, \dots, 0, \dots, 1)\ . \end{aligned} \tag{2.40}$$

[6] We use quotation marks as fields related by (2.36) are not on the same gauge orbit.

To find the topological charge of the KK monopole, eq. (2.38) plus gauge covariance of the field strength allow us to argue that:

$$\begin{aligned} Q &= \frac{1}{32\pi^2}\int d^3x dy G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}\left[A^{KK}(\tilde{V})\right] = \frac{1}{32\pi^2}\int d^3x dy G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}\left[A^{PS}(\tilde{V}')\right] \\ &= \frac{L}{4\pi^2}\int_{S^2_\infty} d^2\sigma^m \mathrm{tr} A_4(\tilde{V}')B^m_{PS} = -n_N\frac{\tilde{V}'L}{2\pi} = -n_N\left(1-\frac{\tilde{V}L}{2\pi}\right), \end{aligned} \tag{2.41}$$

the calculation in complete analogy with (2.30), using the asymptotics of A_4, eq. (2.33) with $\tilde{V}\to\tilde{V}'$, and of $B^m_{PS} = -B^m_{KK}$ of (2.40). Thus, remembering from eq. (2.26) that $I^2_{\mathcal{R}} = -2T(\mathcal{R})Q$, we obtain that for n_N KK monopoles, the topological charge contribution to the index is:

$$I^{2,KK}_{\mathcal{R}}(0) = 2T(\mathcal{R})\, n_N\left(1-\frac{\tilde{V}L}{2\pi}\right). \tag{2.42}$$

The computation of the surface term $I^1_{\mathcal{R}}(0)$ is also simplified by the fact that the asymptotics of the KK monopole solution at infinity are x_4 independent and are, as explained above, the same as those for the PS monopole, except for a switch in the sign of the magnetic field. Thus, despite the fact that in the "bulk" the solution is twisted around S^1, we can still use (2.13) to calculate the surface term contribution. Substituting eqs. (2.40) and (1.11) into (2.13), we obtain for the fundamental representation of SU(N), instead of (2.25):

$$I^{1,KK}_{fund.}(0) = n_N\left(\frac{\tilde{V}L}{2\pi} - \left\lfloor\frac{\hat{v}_N L}{2\pi}\right\rfloor + \left\lfloor\frac{\hat{v}_1 L}{2\pi}\right\rfloor\right). \tag{2.43}$$

Combined with (2.42), this gives for the total index of the KK monopole:

$$I^{KK}_{fund.}(0) = n_N\left(1 - \left\lfloor\frac{\hat{v}_N L}{2\pi}\right\rfloor + \left\lfloor\frac{\hat{v}_1 L}{2\pi}\right\rfloor\right). \tag{2.44}$$

In the case where for all j the holonomies obey $|\hat{v}_j| < \frac{\pi}{L}$, taking into account our ordering of the holonomy (1.11) ($\hat{v}_1 < 0$, $\hat{v}_N > 0$), we have, for $n_N = 1$, that $I^{KK}_{fund} = 0$. Recall from the discussion around eq. (2.32) that in the background of $n_1, n_2, \ldots, n_{N-1}$ monopoles corresponding to the 1st, 2nd, etc., simple roots there are n_{j^*} fermionic zero modes, where j^* is the position of the last negative $\hat{v}_j$ from (1.11). Thus, the combination of a $n_{j^*} = 1$ monopole and an $n_N = 1$ KK monopole have a combined number of zero modes equal to that of a four-dimensional BPST (anti) instanton (one for the fundamental of SU(N)); the sum of their topological charges also adds to minus one.

At this stage, we can also combine (2.31) and (2.44) into a single formula:

$$\begin{aligned} I_{fund.}[n_1, n_2, \ldots, n_{N-1}, n_N] &= I_{fund.}(n_1, n_2, \ldots, n_{N-1}) + I^{KK}_{fund.}(n_N) \\ &= n_N - \sum_{j=1}^{N} n_j\left(\left\lfloor\frac{L\hat{v}_j}{2\pi}\right\rfloor - \left\lfloor\frac{L\hat{v}_{j+1}}{2\pi}\right\rfloor\right). \end{aligned} \tag{2.45}$$

where $L\hat{v}_{N+1} \equiv L\hat{v}_1$.

3 SU(2) with arbitrary representation fermions

The calculation of the index is particularly simple for arbitrary representations of SU(2). Consider, for example, a Weyl fermion in the spin-j representation of SU(2) in the static BPS background. The asymptotic form of the A_4 and magnetic fields are:

$$A_4|_\infty = -v\ (T^3)_j = -v \text{ diag }(j, j-1, \ldots, -j), \qquad B^m|_\infty = \frac{\hat{x}^m}{|x|^2}\ (T^3)_j\ , \tag{3.1}$$

where we set $n_1 = 1$ for simplicity. The index receives contribution from the surface term (2.15) and topological charge (2.26). Instead of (2.15), we now have:

$$I^1_j(0) = -\sum_{m=-j}^{j} m \sum_{p=-\infty}^{\infty} \text{sign}\left(-vm + \frac{2\pi p}{L}\right)\ , \tag{3.2}$$

where the minus sign in the sign-function is because in our convention the holonomy at infinity is $A_4 \simeq -vT^3$. We perform the KK sum in a way similar to (2.24) to obtain:

$$I^1_j(0) = \sum_{m=-j}^{j} -m^2\frac{vL}{\pi} - 2m\left\lfloor -\frac{vmL}{2\pi}\right\rfloor\ . \tag{3.3}$$

For the topological charge contribution, we can use the first line of (2.30) and following the steps that led to (2.29), we obtain:

$$I^2_j(0) = 2T(j)\frac{Lv}{2\pi}\ . \tag{3.4}$$

Recall that for the spin-j representation of SU(2), the Casimir is given by $T(j) = \sum_{m=-j}^{j} m^2 = \frac{1}{3}j(j+1)(2j+1)$. Therefore, summing over the two contributions (3.3) and (3.4) to the index, we find:

$$I_j(0) = \sum_{m=-j}^{j} 2m\left(-\frac{mvL}{2\pi} - \left\lfloor -\frac{mvL}{2\pi}\right\rfloor\right) + 2T(j)\ \frac{Lv}{2\pi} = -\sum_{m=-j}^{j} 2m\left\lfloor -\frac{mvL}{2\pi}\right\rfloor\ . \tag{3.5}$$

The relation between the index for the BPS monopole and KK monopole is also especially simple in SU(2), where there are only two kinds of monopoles; in the spin-j representation the index in the KK monopole background can be obtained by using techniques of the section (2.2), with the result:

$$I^{KK}_j = 2T(j) - I_j \tag{3.6}$$

where I_j is the index of the j-representation in the monopole field and $2T(j)$ is the number of zero modes in a BPST instanton background.

Let the number of monopoles and KK monopoles in a given background be, respectively, n_1 and n_2. The main result of this section is captured in the index and the topological charge formulae:

$$\begin{aligned} I_j[n_1,n_2] &= n_1 I_j + n_2 I_j^{KK} \ = \ n_2 2T(j) - (n_1-n_2)\sum_{m=-j}^{j} 2m\left\lfloor -\frac{mvL}{2\pi}\right\rfloor, \\ Q[n_1,n_2] &= n_1 Q^{BPS} + n_2 Q^{KK} = -n_2 + (n_2-n_1)\frac{vL}{2\pi}\ . \end{aligned} \tag{3.7}$$

We consider now as an example the three lowest representations of SU(2). We already discussed the fundamental representation of SU(N). In the appendix, we give expressions for other SU(N) representations of interest.

Index for the fundamental ($j = 1/2$). We have, from (3.5):

$$I_{1/2}(0) = -\left\lfloor -\frac{vL}{4\pi} \right\rfloor + \left\lfloor \frac{vL}{4\pi} \right\rfloor . \tag{3.8}$$

Begin with the case $0 < v < \frac{4\pi}{L}$, when we obtain $I_{1/2} = 1$. That this is so can be easily verified by explicitly solving the zero mode equation for the Weyl operator D in the PS background [19]. This is also the result of the Callias index theorem on R^3, as expected on physical grounds when L is small and the scale v of SU(2)-breaking is below the KK scale.

Upon increasing v, taking $\frac{4\pi}{L} < v < \frac{8\pi}{L}$, we have $I_{1/2} = 3$. More generally, eq. (3.8) implies that the index jumps by two every time v crosses another $\frac{4\pi}{L}$ threshold. This jump of the index occurs because every time v increases by $\frac{4\pi}{L}$, two zero-mode solutions with nonvanishing KK number become normalizable. This jump of the index can be easily seen explicitly by considering the normalizability of the zero-mode solutions of the $D(A)\psi = 0$ Weyl equation in the static PS background on $S^1 \times R^3$, along the lines of the appendix of ref. [9].

Index for the adjoint ($j = 1$). Now we have from (3.5):

$$I_1(0) = -2\left\lfloor -\frac{vL}{2\pi} \right\rfloor + 2\left\lfloor \frac{vL}{2\pi} \right\rfloor . \tag{3.9}$$

Begin with $0 < v < \frac{2\pi}{L}$, where $I_1(0) = 2$, the well-known value in three dimensions. As we increase $\frac{2\pi}{L} < v < \frac{4\pi}{L}$, we obtain $I_1(0) = 6$. Thus, the index jumps by 4 every time v crosses a KK threshold. Again, this is because as v passes beyond $\frac{2\pi}{L}$ every $L = 0$ normalizable zero mode acquires two more normalizable KK partners.

Index for three-index symmetric tensor ($j = 3/2$). Our final example is the three-index symmetric tensor ($j = 3/2$ of SU(2)). This representation alone is free of a Witten anomaly and gives an example of a chiral four-dimensional theory with interesting non-perturbative dynamics. The index of the representation is $T(3/2) = 5$. For this case (3.5) implies that the index is:

$$I_{3/2}(0) = -3\left\lfloor -\frac{3vL}{4\pi} \right\rfloor - \left\lfloor -\frac{vL}{4\pi} \right\rfloor + 3\left\lfloor \frac{3vL}{4\pi} \right\rfloor + \left\lfloor \frac{vL}{4\pi} \right\rfloor . \tag{3.10}$$

For $0 < v < \frac{4\pi}{3L}$, where $I_{3/2}(0) = 4$, as on R^3. As v increases across the first KK threshold to $\frac{4\pi}{3L} < v < \frac{8\pi}{3L}$, we have $I_{3/2} = 10$—a jump of the index by 6. As v crosses the next threshold $\frac{8\pi}{3L} < v < \frac{4\pi}{L}$, we similarly find that the index jumps by 6, giving $I_{3/2}(0) = 16$. Similarly to the previous cases, the jumps are interpreted as due to more KK-fermion zero modes becoming normalizable as v increases through each threshold.

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4 Interpolating from Callias to APS index

It is useful to put together the results for the index theorem on $R^3 \times S^1$ and see how it interpolates between the Callias index theorem on R^3 and the APS index theorem on R^4. This will also provide a crisp notion of an elementary versus composite topological excitation on $R^3 \times S^1$. In order to study these excitations, it is useful to recall some basic facts about the root system of a Lie algebra and the distinction between the simple root system and affine root system.

For a given Lie algebra, we can construct all roots Δ, positive roots Δ^+, and simple positive roots Δ^0, satisfying $\Delta \supset \Delta^+ \supset \Delta^0$. For example, all roots in Δ^+ can be written as positive linear combinations of simple roots which constitute Δ^0:

$$\Delta^0 = \{\alpha_1, \ldots, \alpha_{N-1}\}\,, \tag{4.1}$$

where α_i are $N-1$ linearly independent simple roots. The simple root system is useful in the discussion of the elementary static monopoles, and the discussion of index theorems on R^3.

On $R^3 \times S^1$, there is an extra monopole, the KK-monopole, which is on the same footing with the monopoles. The existence of this extra topological excitation is significant in multiple ways. For example, as it will be seen below, one can only construct the four dimensional BPST instanton out of the "constituent monopoles" due to the existence of the KK monopole. Incorporating the KK-monopole into the set of "elementary" monopoles also has a simple realization in terms of Lie algebra. There is a unique extended root system (or extended Dynkin diagram) for each Δ^0, which is obtained by adding the lowest root to the system Δ^0:

$$\Delta^0_{\text{aff}} = \Delta^0 \cup \{\alpha_N\} \equiv \{\alpha_1, \ldots, \alpha_{N-1}, \alpha_N\} \tag{4.2}$$

Let $n_1, \ldots, n_N$ denote the number of elementary monopoles whose charges are proportional to $\alpha_1, \ldots, \alpha_N \in \Delta^0_{\text{aff}}$, respectively. The Callias index on R^3, for sufficiently small $|\hat{v}_j L|$, is equal to the index of the Dirac operator on $R^3 \times S^1$ for elementary monopoles with charges taking values in the simple root system Δ_0, i.e.:

$$I_{R^3}[n_1, \ldots, n_{N-1}, 0] = I_{R^3\times S^1}[n_1, \ldots, n_{N-1}, 0]\ . \tag{4.3}$$

This is already demonstrated in obtaining (2.32) from (2.31) by using $|\hat{v}_j L| \leq \pi$.

We now discuss the relation between the APS index for the BPST instanton and the index theorem on $R^3 \times S^1$. The result is:

$$I_{\text{instanton}} = I_{R^3\times S^1}[1, 1, \ldots, 1, 1] = \sum_{i=1}^{N} \ I_{R^3\times S^1}[0, \ldots, \underbrace{1}_{i^{th}}, \ldots, 0] \tag{4.4}$$

The proof of this statement necessitates a convenient rewriting of the index for the "static" (2.31) and "winding" (2.43) solutions. The important technical detail to keep in mind is that for static solutions (2.31), we set $n_0 = n_N = 0$. The index formula for $[n_1, \ldots, n_N]$ monopoles takes the simple form:

$$I_{fund.}[n_1, \ldots, n_N] = n_N - \sum_{j=1}^{N} n_j \left(\left\lfloor \frac{L\hat{v}_j}{2\pi} \right\rfloor - \left\lfloor \frac{L\hat{v}_{j+1}}{2\pi} \right\rfloor \right)\ . \tag{4.5}$$

We also need to show that the topological excitation for which $[n_1, \ldots, n_N] = [1, \ldots, 1]$ corresponds to the BPST instanton. It is obvious that the magnetic charge of such an excitation is identically zero, $\sum_{i=1}^{N} \alpha_i = 0$. We also need to show that the topological charge adds up to the one of a BPST instanton. Using formula (2.30) for static BPS monopoles (and setting $n_0 = n_N = 0$ therein) and (2.41) for the KK monopole, we obtain the topological charge of the excitation:

$$Q[n_1, \ldots, n_N] = -n_N + \sum_{i=1}^{N} n_i \left(\frac{L\hat{v}_i}{2\pi} - \frac{L\hat{v}_{i+1}}{2\pi} \right) . \tag{4.6}$$

For $[n_1, \ldots, n_N] = k[1, \ldots, 1]$, the topological charge is integer valued with no dependence on the specific values of v_i. This is indeed the instanton with winding number k. For index theorem aficionados, eq. (4.4) can also be expressed as:

$$\dim \ker \not{D}_{\text{inst}} - \dim \ker \not{D}^{\dagger}_{\text{inst}} = \sum_{\alpha_i \in \Delta^0_{\text{aff}}} \left(\dim \ker \not{D}_{\alpha_i} - \dim \ker \not{D}^{\dagger}_{\alpha_i} \right) . \tag{4.7}$$

It is evident that the index for Dirac operators on $S^1 \times R^3$ has more refined data than the familiar APS index theorem for instantons on R^4 .

Remark on some special cases. In the derivation of the index theorem for the Dirac operator in the background of a monopole, we used the local axial-current non-conservation (2.5), which is an exact operator identity valid on any four-manifold, and a certain boundary Wilson line $A_4|_\infty$ (1.11). In fact, the index is only well-defined for invertible $A_4|_\infty$. In this case, the corresponding Dirac operator is called a Fredholm operator. An eigenvalue of $A_4|_\infty$ can always be rotated to zero by turning on an over-all Wilson line as in (1.12), which corresponds to a non-Fredholm operator. In those cases, the index for the monopole as well as the η-invariant are not well-defined.

What happens physically as the overall U(1) Wilson line is dialed? In that case, in eq. (4.5), we replace $\hat{v}_j \to \hat{v}_j + \frac{1}{\sqrt{2N}} a_0$ following eq. (1.12). As a_0 is dialed smoothly, the fermionic zero mode will jump from the monopole it is localized into (say, with charge α_{j*}) to a monopole which is nearest neighbor, $\alpha_{j*\pm 1}$, depending on the sign of the a_0. In the mean time, note that the index for the BPST instanton $I_{\text{instanton}} = I_{R^3 \times S^1}[1, 1, \ldots, 1, 1]$ in eq. (4.4) should remain invariant. As the normalizable zero mode jumps from α_{j*} to $\alpha_{j*\pm 1}$, exactly at the value of a_0 where one of the eigenvalues becomes zero, a non-normalizable zero mode appears and the exponential decay of the zero mode wave function is replaced by a power law decay of the three dimensional massless fermion propagator.

5 Remarks on anomalies and induced Chern-Simons terms on $R^3 \times S^1$

Consider a chiral four-dimensional gauge theory compactified on $R^3 \times S^1$. In the limit of zero radius, one expects that a generic theory on R^3 with complex-representation fermions will violate three dimensional parity. This is because the R^3 theory can not be regulated by Pauli-Villars (PV) fields in a simultaneously parity- and gauge-invariant manner.

For example, in the SU(5) theory with left-handed Weyl fermions in the **5** and $\mathbf{10}^*$, compactified to R^3, four-dimensional Lorentz and gauge invariance would forbid mass terms for the fermions, but on R^3 real mass terms are allowed. Real mass terms in three dimensions can be thought of as expectation values of the A_4 (Wilson line) components of background U(1) gauge fields gauging global chiral symmetries. These mass terms are gauge invariant but break three dimensional parity. On R^3, one can regulate the theory in a gauge invariant manner via real-mass Pauli-Villars fields in the **5** and **10***. It is a well-known result [20] that every PV regulator gives rise to a Chern-Simons (CS) term, proportional to the index of the representation (1/2 for **5** and 3/2 for **10***) and to the sign of its mass. Thus, at one loop, the fermion effective action has a parity-violating CS term, whose coefficient is 1 or 2, depending on the chosen relative sign of the two PV mass terms. This CS term does not give rise to a "parity anomaly," which would require the addition of a gauge-noninvariant bare half-integer coefficient CS term, since the integer coefficient assures its invariance under gauge transformations with nontrivial $\pi_3(G)$ (for a brief reminder of the quantization of the CS coefficient, see the footnote in the beginning of section 5.1). However, it gives a topological mass term to the gauge boson. If a bare CS term with integer coefficient is added, the CS coefficient becomes a free parameter of the three dimensional "chiral" gauge theory. When the gauge group is broken to its maximal Abelian subgroup (by an adjoint Higgs field, as in the applications we have in mind) this will give rise to CS terms with quantized coefficients for the various U(1).

5.1 Loop-induced Chern-Simons terms on $R^3 \times S^1$

Now, consider the same theory on the locally four-dimensional background $R^3 \times S^1$. PV regulators with complex masses are not allowed by gauge invariance, while a real mass due to a Wilson-line expectation value is neither local nor Lorentz invariant. Hence, we are led to reconsider the calculation of the CS term, this time on $R^3 \times S^1$. We would like to know whether such a term is generated and what the freedom in the CS coefficient found in the R^3 case corresponds to on $R^3 \times S^1$. Our main interest is in the case when the gauge group is broken to its maximal Abelian subgroup by the nontrivial holonomy on S^1. Thus, consider the loop-induced CS coefficient k_{ab}:

$$S_{CS} = \int d^3x \, \frac{k_{ab}}{8\pi} \, \epsilon_{lim} \, A_l^a \partial_i \, A_m^b \,, \tag{5.1}$$

where a and b run over the Cartan generators of the gauge group.[7] A straightforward loop calculation of k_{ab} in the background holonomy A_4 gives:

$$\begin{aligned} k_{ab} &= \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{\pi^2} \, \mathrm{tr}\, T^a \, \frac{1}{k^2 + (A_4 + \frac{2\pi n}{L})^2} \, T^b \, \frac{(A_4 + \frac{2\pi n}{L})}{k^2 + (A_4 + \frac{2\pi n}{L})^2} \\ &= \mathrm{tr}\, T^a T^b \sum_{n=-\infty}^{\infty} \mathrm{sign}\left(A_4 + \frac{2\pi n}{L}\right), \end{aligned} \tag{5.2}$$

[7]Recall that in the nonabelian case, $S_{CS} = \int d^3x \, \frac{k}{4\pi} \epsilon^{lim} \mathrm{tr}(A_l \partial_i A_m + \frac{2i}{3} A_l A_i A_m)$, where the trace is in the fundamental, and that k is quantized. To see this, let U(x) denote a gauge rotation for which $\pi_3(G)$ is non-trivial, i.e, $\int \frac{1}{24\pi^2} \epsilon^{\nu\lambda\kappa} \mathrm{tr}[U\partial_\nu U^\dagger \; U\partial_\lambda U^\dagger \; U\partial_\nu U^\dagger] \equiv \int \omega(x) \in \mathbb{Z}$. Under a gauge transformation, the variation of the action is given in footnote (5) and yields $S_{CS}(A^U) = S_{CS}(A) + i(2\pi k) \int \omega(x)$, in Euclidean space, showing that gauge invariance of the partition function demands quantization of k.

where a sum over all fermion matter representations is understood in the trace. To obtain the second equality we noted that all generators above are in the Cartan and took the momentum integral, leading to a KK sum identical to the one appearing in (2.15). Finally, we regulate the sum via ζ-function as in the calculation of the η-invariant, and obtain:

$$k_{ab} = \text{tr}\, T^a T^b\, \eta[A_4, 0] = \text{tr}\, T^a T^b \left(1 - 2\frac{LA_4}{2\pi} + 2\left\lfloor \frac{LA_4}{2\pi}\right\rfloor\right) , \tag{5.3}$$

where the function $\lfloor \ldots \rfloor$ is applied to each element of the diagonal matrix A_4. To further understand (5.3), note that if[8] $|A_4| < \frac{\pi}{L}$, we have $1 - 2\frac{LA_4}{2\pi} + 2\left\lfloor \frac{LA_4}{2\pi}\right\rfloor = -2\frac{LA_4}{2\pi} + \text{sign}A_4$, and that after inserting this in (5.3) and using $k_{ab} = k_{ba}$, we find:

$$k_{ab} = -\text{tr}(\{T^a, T^b\}A_4)\frac{L}{2\pi} + \text{tr}(T^a T^b \text{sign}A_4) . \tag{5.4}$$

To understand the meaning of the two terms in (5.4), we now use the decomposition of the sign matrix $\text{sign}(A_4)$ in each representation $\mathcal{R}$ in terms of the unit matrix and Cartan generators:

$$\text{sign}(A_{4\mathcal{R}}) = s^0 1 + \sum_{c=1}^{r} s^c T^c, \quad s^0 = \frac{1}{\dim(\mathcal{R})}\text{tr}_{\mathcal{R}}[\text{sign}(A_4)], \quad s^a = \frac{1}{T(\mathcal{R})}\text{tr}_{\mathcal{R}}[\text{sign}(A_4)T^a] , \tag{5.5}$$

and a similar decomposition for the holonomy A_4 itself:

$$\left.\frac{A_4 L}{2\pi}\right|_{\mathcal{R}} = a^0 1 + \sum_{c=1}^{r} a^c T^c, \quad a^0 = \frac{L}{2\pi \dim(\mathcal{R})}\text{tr}_{\mathcal{R}}[A_4], \quad a^c = \frac{L}{2\pi T(\mathcal{R})}\text{tr}_{\mathcal{R}}[A_4 T^c] . \tag{5.6}$$

After inserting these in (5.4), we find:

$$k_{ab} = \sum_{\mathcal{R}} \left[\text{tr}_{\mathcal{R}} \left(\{T^a, T^b\}T^c\right)(s^c - a^c) + T(\mathcal{R})\delta_{ab}(s^0 - a^0)\right] . \tag{5.7}$$

If A_4 is entirely in the Cartan subalgebra of the gauge group, then $a^0 = 0$. Furthermore, if the sign matrix is traceless ($s^0 = 0$)—which is the case for SU($2N$) theories with a center symmetric background — we find that the CS coefficient on $R^3 \times S^1$ is proportional to the coefficient of the gauge anomaly in four dimensions (recall that the anomaly coefficient for a representation $\mathcal{R}$ is $\text{tr}_{\mathcal{R}}\left(\{T^a, T^b\}T^c\right)$). In this case, we find that for anomaly-free chiral gauge theories in four dimensions there is no loop induced CS term in three dimensions.

It can happen that the sign matrix is not traceless, in which case the only contribution to the CS term is from the second term in (5.4), proportional to tr $\text{sign}A_4$. For example, in anomaly-free SU($2N+1$) gauge theories with an almost center symmetric holonomy, while the first term in (5.4) vanishes, the second term in (5.4) may still be non-zero. In such cases, one can tune a background Wilson line associated with an axial, non-anomalous U(1) to isolate a point where CS-term vanishes (an example of this kind is SU(5) theory with **5** and **10***).

[8]If this condition is not obeyed, the following equations have to be modified accordingly, as was done in the computation of the index.

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In conclusion, we find that the CS coefficient on $R^3 \times S^1$ receives two contributions. The first is a "four-dimensional" one and is given by the first term in (5.4). If the only Wilson lines that are turned on correspond to anomaly-free gauge and global symmetries, the contribution of this term vanishes. On the other hand, turning on Wilson lines corresponding to anomalous symmetries leads to a nonvanishing first term in (5.4)—its origin is in the four-dimensional Wess-Zumino term induced when anomalous background fields are included (the reason the calculation in the nontrivial holonomy phase is so simple is that breaking the gauge symmetry and having massive fermions propagate in the loop allowed us to turn the four dimensional Wess-Zumino term into a local three dimensional CS term). The second, "three-dimensional," contribution [20] is given by the second term in (5.4) and is nonzero only if $\frac{\mathrm{tr}_{\mathcal{R}}\, \mathrm{sign} A_4}{\dim(\mathcal{R})}$ generates an anomalous U(1) symmetry in the four-dimensional theory.

5.2 Excision of topological excitations and remnant Chern-Simons theories

In the beginning of this section, we found that in the three dimensional reduction of a four dimensional chiral theory, there is freedom to have CS terms with quantized coefficients. Is there similar freedom in the theory on $R^3 \times S^1$? The answer can again be seen from (5.4).

In section 5.1, we assumed that the only Wilson lines turned on are those corresponding to the Cartan generators of the gauge group. We are free, however, to turn on Wilson lines of background U(1) fields gauging global chiral symmetries in four dimensions. These Wilson lines do not break the gauge symmetry, but the symmetries they correspond to are usually anomalous, hence we can use eq. (5.4) to infer the CS coefficient induced when they are turned on. It is clear from (5.4) that, generally, the value of the CS coefficient induced in the nonzero holonomy phase by these "flavor" Wilson lines would not correspond to quantized values, unlike in three dimensions. However, recall that turning on Wilson lines for global symmetries is equivalent, by a field redefinition, to imposing non-periodic boundary conditions on the Weyl fermions in $\mathcal{R}$,[9]

$$\psi(x, y+L)_{\mathcal{R}} = e^{i\alpha_{\mathcal{R}}} \psi(x,y)_{\mathcal{R}}\,, \quad \alpha_{\mathcal{R}} = A_4^{\mathcal{R}} L\ . \tag{5.8}$$

Consequently, we find from (5.4), assuming that only a U(1) Wilson line, $A_4^{\mathcal{R}}$, is turned on, that:

$$k_{ab} = \delta_{ab} \sum_{\mathcal{R}} \left(-\frac{2\alpha_{\mathcal{R}}\, T(\mathcal{R})}{2\pi} + T(\mathcal{R})\ \mathrm{sign}\ \alpha_{\mathcal{R}} \right) \equiv \delta_{ab}\, k(\alpha). \tag{5.9}$$

The induced CS term is, therefore,

$$S_{CS} = \frac{k(\alpha)}{4\pi} \int_{R^3} \epsilon^{\nu\lambda\kappa}\ \mathrm{tr} \left(A_\nu \partial_\lambda A_\kappa + \frac{2i}{3} A_\nu A_\lambda A_\kappa \right)\ . \tag{5.10}$$

Note that in the case of anomalous-U(1) Wilson lines, the boundary conditions (5.8) would correspond to a symmetry of the action and measure of the theory — hence be admissible as boundary conditions — only if the Wilson lines take quantized values,

$$2T(\mathcal{R})\alpha_{\mathcal{R}} = 2\pi n,\ \ n \in \mathbb{Z}\,, \tag{5.11}$$

[9] For complex representation Dirac fermions, these "chirally-twisted" boundary conditions can also be rewritten as $\Psi(x, y+L) = e^{i\alpha_{\mathcal{R}}\gamma_5}\Psi(x,y)$.

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implying that admissible boundary conditions for fermions are quantized (such that the 't Hooft vertex is invariant). Thus, the coefficients of the induced CS terms in this case also take quantized values.

Note that the phase structure of gauge theory — massive versus perturbatively massless photons — is affected by turning on such discrete Wilson lines. Since the values are quantized, the one-loop potential for the Wilson line (Casimir energies) should not effect them (discrete Wilson lines are known to appear in string theory, for example as disconnected components on the moduli space of D-branes [21]). Moreover, at nonzero k, the finite action monopole solutions (or other topological excitations, such as magnetic bions pertinent to gauge theories on $S^1 \times R^3$) which would render the gauge fluctuations massive nonperturbatively do not exist; see, e.g., [22]. In this sense, the two types of possible mass terms for gauge fluctuations, parity odd topological CS mass and parity even magnetic monopole or bion induced mass do not mix.

To summarize, since the chiral anomalous U(1) current is parity odd, the response of gauge theory is to produce a non-gauge invariant CS term at generic values of the background Wilson line. Only at admissible (discrete set of) boundary conditions for fermions, the induced CS term is gauge invariant and sensible, and a parity odd mass term is generated for the gauge theory. At these points, the finite action topological excitation are excised from the gauge theory. If no anomalous U(1) is turned on, the photon is massless to all orders in perturbation theory, and a parity even mass term can be induced non-perturbatively via topological excitations with zero index, either elementary or composite.

The notion of the disconnected components of the gauge theory "moduli space" may find interesting applications both in physics and mathematics. First, we formulate a QCD-like gauge theory on $\mathcal{M}_3 \times S_1$ where $\mathcal{M}_3$ is some three-manifold of arbitrary size, and S_1 is small. Then, we impose admissible boundary conditions on the (say) adjoint Weyl fermion[10] by using a "chiral twist" (5.8) obeying (5.11), by taking $\alpha = \frac{2\pi n}{2N}$, and assuming that n is a positive integer (this is to say that only a $\mathbb{Z}_{2N}$ is a anomaly-free remnant of the $\mathrm{U}(1)_A$ chiral rotation, and allowed as boundary condition). Integrating out all the heavy KK-modes along the S_1 circle induces, among other operators, the CS-term (5.10) with coefficient given by (5.9) and equal to $k(\alpha) = N - n$. This means that a CS-term does not get induced for strictly periodic and anti-periodic (thermal) boundary conditions. Otherwise, we expect that the long distance dynamics of these disconnected components of the "moduli space" of QCD-like theories is described by topological CS-theory on $\mathcal{M}_3$. This means that the theory is gapped, and is in a topologically ordered Chern-Simons phases. Up to our knowledge, this is the first derivation of CS-theory and topological phases from QCD-like dynamics. We will pursue this direction in subsequent work.

[10]That this theory is actually SYM plays no role, as one can similarly consider multi-adjoint theories. The discussion can also be generalized to Dirac fermions in complex, two-index representations. For $\mathcal{R} = \{\mathrm{AS}, \mathrm{S}, \mathrm{BF}, \mathrm{F}\}$ representations, there are respectively, $2T(\mathcal{R})$ disconnected components (B.8), and for $2T(\mathcal{R}) - 1$ of them, the long distance physics reduces to pure Chern-Simons theory. In particular, for QCD with one fundamental fermions, there is no admissible boundary condition for which the infrared physics reduces to CS-theory.

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Note added. While completing this paper, a new preprint [26] appeared, which discusses the relation between the R-symmetry-twisted boundary conditions (which is a chiral-twist in our nomenclature) and CS theories in supersymmetric $\mathcal{N}=4$ SYM theory. Our discussion in section 5 has some overlaps with the discussion therein.

A Another calculation of the η-invariant

Here we give an alternative computation of $\eta_j[0]$ of eq. (2.16). We now use the form [23, 24]:

$$\eta[0] = \frac{1}{\pi} \lim_{m\to\infty} \sum_{\lambda} \mathrm{Im}\ln\frac{\lambda+im}{\lambda-im} = \frac{1}{\pi}\lim_{m\to\infty} \mathrm{Im}\ln\det\frac{h+im}{h-im}\,, \tag{A.1}$$

which holds because:

$$\lim_{m\to\infty}\frac{\lambda+im}{\lambda-im} = \lim_{m\to\infty} e^{i2\mathrm{Arctan}\left(\frac{m}{\lambda}\right)} = e^{i\pi\,\mathrm{sign}\lambda}\,,$$

and the branch of the logarithm is defined so that $\ln e^{i\phi} = i\phi$ (zero eigenvalues λ are assumed to not occur; if they do the formula (A.1) is ambiguous and needs to be modified [24]).

Recall from the discussion in paragraph above eq. (2.16) that h is the one-dimensional "massive Dirac operator" $i\frac{d}{dy}+\hat{v}$ whose eigenvalues change sign under the combined $y\to -y$, $\hat{v}\to-\hat{v}$ transformation. Together with (A.1) (or (A.2)) this implies that the spectral asymmetry (A.1) flips sign under $\hat{v}\to-\hat{v}$. For our operator, $\lambda = \frac{2\pi n}{L}+\hat{v}$, so we have:

$$\begin{aligned}\eta[0] &= \frac{1}{\pi}\lim_{m\to\infty}\sum_{n=-\infty}^{\infty}\mathrm{Im}\ln\frac{\frac{2\pi n}{L}+\hat{v}+im}{\frac{2\pi n}{L}+\hat{v}-im} = \frac{1}{\pi}\lim_{m\to\infty}\sum_{n=-\infty}^{\infty}\mathrm{Im}\ln\frac{n+a+im}{n+a-im}\,,\\ a &\equiv \frac{L\hat{v}}{2\pi} - \left\lfloor\frac{L\hat{v}}{2\pi}\right\rfloor \subset (0,1)\,,\end{aligned} \tag{A.2}$$

where in the first line we trivially rescaled m and the second line means that a is taken to be in the interval $(0,1)$, which is always possible to achieve by re-labelling the sum over KK modes. We note that the region $(0,1)$ is the fundamental region, as η is well-defined and smooth for all points (as opposed to the $(-1/2,+1/2)$ region which includes a singular point $a=0$). The computation of $\eta[0]$ is simplified by computing the derivative of (A.2) wrt a. Integration to recover the a-dependent part is then trivially done (note that an a-independent constant in $\eta[0]$ would be irrelevant, since a $\hat{v}_j$-independent term in $\eta_j[0]$ does not contribute to the sum in (2.31); furthermore it is prohibited by the "parity"-odd nature of $\eta[0]$). The derivative of (A.2) wrt a is now given by a convergent sum, which is:

$$\frac{d\eta[0]}{da} = -\frac{1}{\pi}\lim_{m\to\infty} 2m\sum_{n=-\infty}^{\infty}\frac{1}{(n+a)^2+m^2} \equiv -\frac{2}{\pi}\lim_{m\to\infty} m\,F(1,a,m)\,, \tag{A.3}$$

where the function $F(1,a,m)$ is implicitly defined by the last equality and is computed, e.g., in eq. (81) of [25], $F(1,a,m) = \frac{\sqrt{\pi}}{m}(\sqrt{\pi} + 4\sum_{p=1}^{\infty}(\pi p m)^{\frac{1}{2}}\cos(2\pi p a)K_{\frac{1}{2}}(2\pi p m))$. When $m \to \infty$, only the first term in $F(1,a,m)$ survives, $\lim_{m\to\infty} mF(1,a,m) = \pi$, thus $\frac{d\eta[0]}{da} = -2$, which determines $\eta_j[0]$ up to an integration constant:

$$\eta_j[0] = -2a_j + c = -2\left(\frac{L\hat{v}_j}{2\pi} - \left\lfloor\frac{L\hat{v}_j}{2\pi}\right\rfloor\right) + c. \tag{A.4}$$

This periodic (in $\hat{v}_j$) result can be made "parity"-odd by taking $c = 1$, giving back (2.24).

B Index for higher representation fermions

R^3. The Callias index theorem on R^3 can be obtained by restricting the sum in (2.15) to $p = 0$.

$$\begin{aligned}\text{Fund.}: \qquad I_F(n_1,\ldots,n_{N-1}) &= -\frac{1}{2}\sum_{i=1}^{N} \operatorname{sign}(\hat{v}_i)(n_i - n_{i-1}) \\ &= -\frac{1}{2}\sum_{i=1}^{N-1} n_i\Big[\operatorname{sign}(\hat{v}_i) - \operatorname{sign}(\hat{v}_{i+1})\Big] = n_{j*}\ .\end{aligned} \tag{B.1}$$

where $\hat{v}_{j*} < 0 < \hat{v}_{j^*+1}$. Since there is no axial anomaly in $d = 3$, there is no other contribution to the index and this is the final result. A better way to express (B.1), which is easily generalizable to arbitrary representation of the gauge group SU(N) is:

$$I_F(n_1,\ldots,n_{N-1}) = -\mathrm{tr}[\operatorname{sign}(A_4)\cdot\hat{B}]\ . \tag{B.2}$$

where $\operatorname{sign}(A_4)$ is the sign matrix and $\hat{B} = \sum_{i=1}^{N-1} n_i\,(\alpha_i\mathbf{H})$ is the space independent part of eq. (1.17). For an arbitrary representation $\mathcal{R}$, this formula generalizes as:

$$I_{\mathcal{R}}(n_1,\ldots,n_{N-1}) = -\mathrm{tr}_{\mathcal{R}}[\operatorname{sign}(A_4)\cdot\hat{B}]\ . \tag{B.3}$$

Our main interest is in fermionic matter in two index representations of gauge group, namely, adjoint, antisymmetric (AS), symmetric (S) of SU(N) and bi-fundamental (BF) representation of SU(N) $\times$ SU(N). For the adjoint, the index is:

$$\begin{aligned}\text{Adjoint}: \qquad I_{\text{Adj}}(n_1,\ldots,n_{N-1}) &= -\frac{1}{2}\sum_{i,j=1}^{N}\operatorname{sign}(\hat{v}_i - \hat{v}_j)\Big[(n_i - n_{i-1}) - (n_j - n_{j-1})\Big] \\ &= -\sum_{j=1}^{N-1}\sum_{i=1}^{N-1} n_i\Big[\operatorname{sign}(\hat{v}_i - \hat{v}_j) - \operatorname{sign}(\hat{v}_{i+1} - \hat{v}_j)\Big] \\ &= \sum_{j=1}^{N-1} 2n_j\ .\end{aligned} \tag{B.4}$$

This means that in the background of each elementary monopole, there are two fermionic zero modes. For the other two-index representations, the expressions are:

$$\begin{aligned}
\text{BF}: \quad & I_{BF}(n^1_1,\ldots,n^1_{N-1},n^2_1,\ldots,n^2_{N-1}) = -\frac{1}{2}\sum_{i,j=1}^{N}\text{sign}(\hat{v}^1_i-\hat{v}^2_j)\Big[(n^1_i-n^1_{i-1})-(n^2_j-n^2_{j-1})\Big]\,,\\
\text{AS}: \quad & I_{AS}(n_1,\ldots,n_{N-1}) = -\frac{1}{2}\sum_{i>j}^{N}\text{sign}(\hat{v}_i+\hat{v}_j)\Big[(n_i-n_{i-1})+(n_j-n_{j-1})\Big]\,,\\
\text{S}: \quad & I_{S}(n_1,\ldots,n_{N-1}) = -\frac{1}{2}\sum_{i\geq j}^{N}\text{sign}(\hat{v}_i+\hat{v}_j)\Big[(n_i-n_{i-1})+(n_j-n_{j-1})\Big]\,. \qquad \text{(B.5)}
\end{aligned}$$

It is more convenient to express the index for AS/S representations as:

$$\begin{aligned}
I_{AS/S} &= -\frac{1}{4}\sum_{i,j=1}^{N}\text{sign}(\hat{v}_i+\hat{v}_j)\Big[(n_i-n_{i-1})+(n_j-n_{j-1})\Big] \pm \sum_{i}^{N}\text{sign}(2\hat{v}_i)(n_i-n_{i-1})\;.\\
&= -\frac{1}{2}\sum_{i,j=1}^{N-1}n_i\Big[\text{sign}(\hat{v}_i+\hat{v}_j)-\text{sign}(\hat{v}_{i+1}+\hat{v}_j)\Big] \pm \sum_{i}^{N-1}n_i\Big[\text{sign}(2\hat{v}_i)-\text{sign}(2\hat{v}_{i+1})\Big] \qquad \text{(B.6)}
\end{aligned}$$

$\boldsymbol{R^3\times S^1}$**.** This formulae can be straightforwardly generalized to $R^3\times S^1$ by repeating our derivations for the fundamental. The non-integer contributions to the index from the topological charge cancel the corresponding non-integer part of the η-invariant, yielding a general expression; further, if the definition of $\hat{B}$ is extended to include the affine root (1.14), $\hat{B}=\sum_{i=1}^{N}n_i\,(\alpha_i\mathbf{H})$, with $n_1,\ldots,n_{N-1}$ are the monopole numbers of the background and n_N—the KK-monopole number, this equation can be also extended to also include the KK monopole:

$$I_{\mathcal{R}}[n_1,\ldots,n_N] = 2T(\mathcal{R})n_N - \text{tr}_{\mathcal{R}}\left\lfloor\frac{A_4L}{2\pi}\right\rfloor\cdot\hat{B}\,. \qquad \text{(B.7)}$$

The second term in (B.7) follows from (2.13), (2.25) by simply extending the definition of $\hat{B}$ to include the affine-root monopole (and by dropping the non-integer terms) while the first term is due to the (negative) integer topological charge contribution to the index of the KK monopole. For reference,

$$2T(\mathcal{R}) = \{1,2N,N,N+2,N-2\}, \qquad \text{for } \mathcal{R}=\{\text{F},\text{Adj},\text{BF},\text{S},\text{AS}\} \qquad \text{(B.8)}$$

for a single Weyl fermion. Note that

$$I_{\mathcal{R},\text{instanton}} = I_{\mathcal{R},R^3\times S^1}[1,1,\ldots,1,1] = 2T(\mathcal{R}) \qquad \text{(B.9)}$$

is just the APS index for an BPST instanton.

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